Edge colouring of multigraphs and the Tashkinov trees method

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An *edge colouring* of a multigraph G is a function that assigns a colour to each edge of G, such that no two edges sharing a vertex get the same colour. The *chromatic index* $\chi'(G)$ is the smallest k for which there exists an edge colouring of G using k colours. It is clear that the maximum degree $\Delta(G)$ is a lower bound for $\chi'(G)$ for every loopless multigraph G. The classical upper bounds for $\chi'(G)$ are $\chi'(G) \leq 3\Delta(G)/2$ (Shannon's Theorem, 1949) and $\chi'(G) \leq \Delta(G) + \mu(G)$ (Vizing's Theorem, 1964), where $\mu(G)$ denotes the maximum edge multiplicity of G. Thus for simple graphs (i.e. those satisfying $\mu(G) = 1$) there are only two possible values for $\chi'(G)$, namely $\Delta(G)$ and $\Delta(G) + 1$, and by Holyer's classical theorem it is NP-hard to distinguish between them.

In contrast with the case of simple graphs, edge colouring in multigraphs is still not well-understood. In particular, what properties of a multigraph G determine where $\chi'(G)$ lies within the range $[\Delta(G), \min\{\Delta(G) + \mu(G), 3\Delta(G)/2\}]$? A possible answer to this question is the focus of a famous open problem posed by Goldberg and (independently) Seymour in the 1970's. It states that if a loopless multigraph G does not have an edge colouring with $\Delta(G) + 1$ colours, then it contains a natural obstruction consisting of a vertex subset that induces too many edges to be coloured with $\Delta(G) + 1$ colours. A proof of this conjecture due to Chen, Jing and Zang has appeared recently but has yet to be verified.

In this course we will study edge colouring in multigraphs, with an emphasis on the method of *Tashkinov trees*. A sophisticated generalization of the method of alternating paths, this technique was introduced by Tashkinov to address questions such as the Goldberg-Seymour problem. In particular we will see the theorem of Tashkinov that forms the foundation of the method, and how to apply it to obtain some of the many important results for multigraph edge colouring for which it has been used. Moreover, these applications typically come with efficient algorithms, since the proof of Tashkinov's Theorem is algorithmic.

All are welcome to attend. No prior knowledge of edge colouring will be assumed.